# MEAN VALUES OF $L$–FUNCTIONS AND SYMMETRY

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Abstract. Recently Katz and Sarnak introduced the idea of a symmetry group attached to a family of $L$–functions, and they gave strong evidence that the symmetry group governs many properties of the distribution of zeros of the $L$–functions. We consider the mean–values of the $L$–functions and the mollified mean–square of the $L$–functions and find evidence that these are also governed by the symmetry group. We use recent work of Keating and Snaith to give a complete description of these mean values. We find a connection to the Barnes–Vignéras $\Gamma_2$–function and to a family of self–similar functions.

## 1. Introduction

Katz and Sarnak [KS] have introduced the idea of a family of $L$–functions with an associated symmetry type. The symmetry type has been shown to govern the distribution and spacing of zeros in the function field case [KS2], and there is strong numerical evidence [Ru] that it governs the behavior of zeros in more general cases. The paper [ILS] also shows clear evidence that the symmetry type governs the distribution of low–lying zeros of a wide variety of $L$–functions. In this paper we give evidence that the symmetry type of a family of $L$–functions governs the behavior of mean values of the $L$–functions.

The most well–understood mean values are the $2k$th moments of the Riemann $\zeta$–function. We will describe that case in detail, and then compare this to the examples provided by the families of Katz and Sarnak.

It is a folklore conjecture that for every $k \geq 0$ there is a constant $c_k$ such that

$$I_k(T) := \frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k}\ dt \sim c_k \log^{k^2} T$$

as $T \to \infty$. Classically, Hardy and Littlewood [HL] proved that

$$I_1(T) \sim \log T,$$

and Ingham [I] showed

$$I_2(T) \sim \frac{1}{2\pi^2}\log^4 T,$$

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so $c_0 = c_1 = 1$ and $c_2 = 1/2\pi^2$. No other mean values of the $\zeta$-function have been established.

More information can be given about $c_k$, for Conrey and Ghosh [CG1] showed (assuming the Riemann Hypothesis) that for all $k \ge 0$,

$$I_k \ge (1+o(1))\frac{a_k}{\Gamma(1+k^2)}\log^{k^2} T,$$

where $a_k$ is an arithmetic constant given by

$$a_k = \prod_p (1-1/p)^{k^2} \sum_{j=0}^{\infty} \frac{d_k(p^j)^2}{p^j}. \tag{1.1}$$

Here $d_k(n)$ is the $n$th coefficient in the Dirichlet series for $\zeta(s)^k$; it is multiplicative and is given by

$$d_k(p^j) = \frac{\Gamma(k+j)}{\Gamma(k)j!}.$$

Gonek [G] extended the above result to include all $k > -\frac{1}{2}$.

Conrey and Ghosh [CG3] established properties of $a_k$. These included that $a_k$, regarded as a function of $k$, is entire of order 2, satisfies a symmetry $a_k = a_{1-k}$, and has all of its zeros on the line $\Re k = \frac{1}{2}$. In addition, all of the zeros of the derivative $\frac{d}{dk}a_k$ have real parts equal to $\frac{1}{2}$ with two exceptions $a_0' = a_1' = 0$. Conrey and Gonek [CGo] give an asymptotic formula for $\log a_k$ as $k \to \infty$. This formula is relevant to understanding the extremely large values of $\zeta(s)$.

Conrey and Ghosh [CG3] defined the number $g_k$ implicitly by the (conjectural) formula

$$I_k(T) \sim g_k \frac{a_k}{\Gamma(1+k^2)}\log^{k^2} T. \tag{1.2}$$

The quantity $g_k$ is natural in the following sense. All approaches to mean value theorems for Dirichlet series have relied on techniques from the theory of Dirichlet polynomials. The main tool there is the mean value theorem of Montgomery and Vaughan [MV]:

$$\int_0^T \left|\sum_{n=1}^N a_n n^{it}\right|^2 dt = \sum_{n=1}^N (T+O(n))|a_n|^2.$$

To use this to obtain asymptotic formulae it is usually necessary to have $N \ll T$. Thus, it is natural in some sense to measure the mean square of $\zeta(s)^k$ against the mean value of a Dirichlet polynomial approximation to $\zeta(s)^k$ using a polynomial of length $T$. It is easy to show that

$$\frac{1}{T}\int_0^T \left|\sum_{n\le T} \frac{d_k(n)}{n^{1/2+it}}\right|^2 dt \sim \frac{a_k}{\Gamma(1+k^2)}\log^{k^2} T.$$

Thus, $g_k$ is a measure of "how many polynomials of length $T$ are needed to capture the mean square of $\zeta(s)^k$."

The classical results of Hardy and Ingham can be phrased as $g_1 = 1$ and $g_2 = 2$. Recently, Conrey and Ghosh [CG2] made the conjecture that $g_3 = 42$. Still more recently, Conrey and Gonek [CGo] conjectured that $g_4 = 24024$. The conjectures for $g_3$ and $g_4$ are based on Dirichlet polynomial techniques. With current methods it is probably not possible to use those techniques to conjecture $g_k$ for larger values of $k$. We will return to the function $g_k$ after giving a general discussion of mean values and a description of the situation for the families of $L$–functions of Katz and Sarnak.

We will now describe the situation for some of the families of Katz and Sarnak. The mean values will be of the shape

$$\frac{1}{\mathcal{Q}^*} \sum_{\substack{f\in\mathcal{F}\\ c(f)\le\mathcal{Q}}} V(L_f(\tfrac{1}{2}))^k \sim g_k \frac{a_k}{\Gamma(1+B(k))} (\log \mathcal{Q}^A)^{B(k)}. \tag{1.3}$$

The $L$–functions are normalized to have a functional equation $s \leftrightarrow 1-s$, so $L_f(\frac{1}{2})$ is the "critical value." Here we think of the family $\mathcal{F}$ as being partially ordered by "conductor" $c(f)$, with $\mathcal{Q}^*$ the number of elements with $c(f) \le \mathcal{Q}$. We set $V(z) = z$ or $|z|^2$ depending on the symmetry type of the family. The parameters $g_k$ and $B(k)$ depend only on the symmetry type and are integral for integral $k$, with $g_1 = 1$. The parameter $a_k$ depends on the family in a natural way and is similar to the case of the $\zeta$–function. The parameter $A$ depends both on the symmetry type and the functional equation satisfied by the elements in the family (specifically, it depends on the degree of the functional equation in the relevant parameter). Examples are given below.

One of the symmetry types described by Katz and Sarnak is denoted O, for "orthogonal." Examples of families with this symmetry type conjecturally include

a) the $L$–functions $L_f(s)$ associated with cusp forms $f \in S_m(\Gamma_0(1))$ of weight $m$ for the full modular group,
b) the $L$–functions $L_f(s)$ with $f \in S_2(\Gamma_0(N))$ of weight 2 for the Hecke congruence group $\Gamma_0(N)$,
c) the twisted $L$–functions $L(s, \mathrm{sym}^{2\ell+1}(f) \otimes \chi_d)$ where $f$ is a self–dual cuspidal automorphic form on $GL_2$ and $\chi_d$ is a quadratic Dirichlet character mod $|d|$, provided that the Fourier coefficients of $f$ have a Sato–Tate distribution, and
d) the twisted $L$–functions $L_f(s, \chi_d)$ where $f$ is a self–dual cuspidal automorphic form on $GL_m$ for some $m$, provided that the symmetric square $L$–function of $f$ does not have a pole at $s = 1$.

To illustrate how the symmetry type is related to the mean values of the $L$–functions in the family, we note the following conjectures.

$$\frac{1}{m^*} \sum_{f\in S_m(\Gamma_0(1))} L_f(\tfrac{1}{2})^k \sim g_k \frac{a_k}{\Gamma(1+\frac{1}{2}k(k-1))} (\log m^{\frac{1}{2}})^{\frac{1}{2}k(k-1)}$$

where $m^*$ is the cardinality of $S_m$, and

$$\frac{1}{N^*} \sum_{f\in S_2(\Gamma_0(N))} L_f(\tfrac{1}{2})^k \sim g_k \frac{a_k}{\Gamma(1+\frac{1}{2}k(k-1))} (\log N^{\frac{1}{2}})^{\frac{1}{2}k(k-1)}$$

where $N^*$ is the cardinality of $S_2(\Gamma_0(N))$, and also

$$\frac{1}{D^*}\sum_{|d|\le D} L(\tfrac12, \mathrm{sym}^{2\ell+1}(f)\otimes\chi_d)^k \sim g_k \frac{a_k}{\Gamma(1+\frac12 k(k-1))}(\log D^{\frac12})^{\frac12 k(k-1)},$$

where $D^*$ is the number of quadratic characters with conductor not exceeding $D$.

Note: these families can be further broken into the even forms and the odd forms, each of which is approximately half of the family. The average over the odd forms is identically zero because the associated $L$–functions vanish at the center of the critical strip.

We were unable to locate a reference for these conjectures. The above formulas were found, for example, using the Petersson formula, and are based on results found in [D][DFI] and [KMV]. From those papers we obtain $g_1 = 1$, $g_2 = 2$, $g_3 = 2^3$ and $g_4 = 2^7$.

In the notation of (1.3), the families with symmetry type O have $V(z) = z$ and $B(k) = \frac12 k(k-1)$. For the remaining parameter we have $A = \mathcal{A}$, where $\mathcal{A}$ is the degree to which the parameter $\mathcal{Q}$ occurs in the functional equation. For example, the first two families above satisfy a functional equation of the form

$$\Phi(s) = \left(\frac{N^{\frac12}}{2\pi}\right)^s \Gamma\left(s+\frac{m-1}{2}\right) L_f(s) = \varepsilon\bar{\Phi}(1-s).$$

That functional equation has degree $\frac12$ in both $N$ and $m$ aspect, so in the corresponding mean value we have $A = \frac12$.

Another symmetry type considered by Katz and Sarnak is denoted Sp, for "Symplectic." Examples conjecturally are:

e) Dirichlet $L$–functions $L(s,\chi_d)$, where $\chi_d$ is a quadratic Dirichlet character mod $|d|$,
f) the symmetric square $L$–functions $L(s, \mathrm{sym}^2(f))$ associated with $f \in S_m(\Gamma_0(1))$,
g) the twisted $L$–functions $L(s, \mathrm{sym}^{2\ell}(f)\otimes\chi_d)$ where $f$ is a self–dual cuspidal automorphic form on $GL_2$, provided that the Fourier coefficients of $f$ have a Sato–Tate distribution, and
h) the twisted $L$–functions $L_f(s,\chi_d)$ where $f$ is a self-dual cuspidal automorphic form on $GL_m$ for some $m$, provided the symmetric square $L$–function of $f$ has a pole at $s = 1$.

The conjectured mean value in case e) is:

$$\frac{1}{D^*}\sum_{|d|\le D} L(\tfrac12,\chi_d)^k \sim g_k \frac{a_k}{\Gamma(1+\frac12 k(k+1))}(\log D^{\frac12})^{\frac12 k(k+1)},$$

where $D^*$ is the number of quadratic characters with conductor not exceeding $D$. This conjecture is based on work of Jutila [J] and Soundararajan [S]. In this case we have $g_1 = 1$, $g_2 = 2$, $g_3 = 2^4$, and it is conjectured that $g_4 = 3\cdot 2^8$.

In case f) we have the conjecture

$$\frac{1}{m^*}\sum_{f\in S_m(\Gamma_0(1))} L(\tfrac12, \mathrm{sym}^2(f))^k \sim g_k \frac{a_k}{\Gamma(1+\frac12 k(k+1))}(\log m^{\frac12})^{\frac12 k(k+1)},$$

where $m^*$ is the cardinality of $S_m$. The cases g) and h) look just like e) above.

In the notation of (1.3), families with symmetry type Sp have $V(z) = z$ and $B(k) = \frac{1}{2}k(k+1)$. The parameter $A$ for the Sp families is determined in exactly the same way as for the O families.

A third symmetry type described by Katz and Sarnak is denoted U, for "Unitary." An example of a family with this symmetry type is

i) $L(s,\chi)$ for $\chi$ a character mod $q$

The conjectured mean value in this case is

$$\frac{1}{Q^*}\sum_{|q|\le Q}\sum_{\chi \bmod q}|L(\tfrac{1}{2},\chi)|^{2k} \sim g_k\frac{a_k}{\Gamma(1+k^2)}(\log Q)^{k^2},$$

where the inner sum is over characters mod $q$, and $Q^*$ is the number of characters with conductor at most $Q$.

For families with symmetry type U we have $V(z) = |z|^2$ and $B(k) = k^2$. For the remaining parameter we have $A = 2\mathcal{A}$, where $\mathcal{A}$ is the degree to which the parameter $\mathcal{Q}$ appears in the functional equation. As the above example suggests, we can think of the Riemann $\zeta$–function as forming its own Unitary family, where the mean values of the $\zeta$–function correspond to averages of special values of the family $\{\zeta(\frac{1}{2}+it)\}_{t\in\mathbb{R}}$.

We end this discussion by giving an example of the parameter $a_k$ in the above formulas. The following is the $a_k$ associated to with family of real Dirichlet $L$–functions $L(s,\chi_d)$, which has symmetry type Sp:

$$a_k = \prod_p \frac{\left(1-\frac{1}{p}\right)^{\frac{k(k+1)}{2}}}{\left(1+\frac{1}{p}\right)}\left(\frac{1}{2}\left(\left(1+\frac{1}{\sqrt{p}}\right)^{-k}+\left(1-\frac{1}{\sqrt{p}}\right)^{-k}\right)+\frac{1}{p}\right).$$

While all of the above mean values are conjectural (except for certain small values of $k$), the various parameters in the formulas are fairly well understood, except for the constant $g_k$. Recently, Keating and Snaith [KeSn] have used techniques from random matrix theory to obtain conjectures for $g_k$ for the symmetry types given above. This is described in the next section.

In this paper we study the functions $g_k$ in detail. We also report on calculations of the mean square of the $L$–function times a "mollifier" for each of the symmetry types described above. The results on $g_k$ are summarized in the next section, and the results on mollifiers are presented in the following section. The remainder of the paper is devoted to results related to the various properties of the $g_k$ functions.

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## 2. Statement of Results

The constants $g_k$ occurring in the conjectured mean values of these $L$–functions is the most mysterious aspect of these mean values. Conjectures for $g_k$ have recently been given, which we now describe.

The methods used to obtain conjectures for $g_k$ are based on random matrix theory. This approach to the study of $L$–functions began with Montgomery's work [M] on the pair correlation of zeros of the Riemann $\zeta$–function, and has been fruitful for establishing both rigorous and conjectural results about $L$–functions. The underlying idea is the conjecture that the zeros of $L$–functions are distributed on the critical line like the eigenvalues of matrices from the Gaussian Unitary Ensemble of large random Hermitian matrices. This is referred to as the "GUE" conjecture. This conjecture has been corroborated, to a large extent, by the extensive computations of Odlyzko [O]. Heuristic explanations for the GUE conjecture have been given by Bogolmony and Keating [BK]. Now Keating and Snaith [KeSn] have announced conjectures for $g_k$ that have been obtained by techniques from this theory.

Keating and Snaith's conjecture in the Unitary case is that

$$g_{\lambda,U} = \Gamma(1+B_U(\lambda)) \lim_{N\to\infty} N^{-\lambda^2} \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\lambda)}{\Gamma(j+\lambda)^2}. \tag{2.1}$$

where $B_U(\lambda)=\lambda^2$. The conjecture in the Orthogonal case is

$$g_{\lambda,O} = \Gamma(1+B_O(\lambda)) \lim_{N\to\infty} 2^{2N\lambda} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(j-\frac{1}{2}+\lambda)}{\Gamma(N+j-1+\lambda)\Gamma(j-\frac{1}{2})} \tag{2.2}$$

where $B_O(\lambda)=\frac{1}{2}\lambda(\lambda-1)$. And the conjecture in the Symplectic case is

$$g_{\lambda,Sp} = \Gamma(1+B_{Sp}(\lambda)) \lim_{N\to\infty} 2^{2N\lambda} \prod_{j=1}^{N} \frac{\Gamma(N+j+1)\Gamma(j+\frac{1}{2}+\lambda)}{\Gamma(N+j+1+\lambda)\Gamma(j+\frac{1}{2})}, \tag{2.3}$$

with $B_{Sp}(\lambda)=\frac{1}{2}\lambda(\lambda+1)$.

We have recently learned that Brézin and Hikami [BH] have independently obtained the above conjectures in the Sp and O cases.

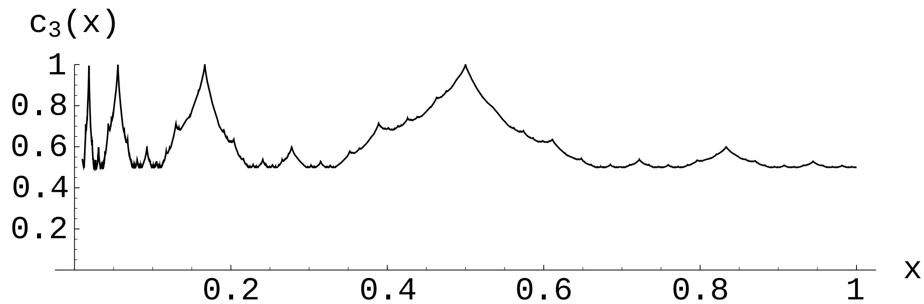
It is of key importance that the above formulas agree with the known and conjectural values of $g_k$ given earlier. These formulas for $g_\lambda$ suggest that it would be natural to absorb the factor $\Gamma(1+B(\lambda))$ in to the definition of $g_\lambda$. However, doing so would obscure the arithmetic origins of the parameters in the mean values, and we would lose the significant fact that $g_k$ is integral for $k$ a positive integer.

In this paper, we develop the properties of the Keating–Snaith constants $g_\lambda$. We show that $g_\lambda/\Gamma(1+B(\lambda))$ is a nonvanishing meromorphic function of order 2. We give a complete description of its pole locations, and we express $g_\lambda/\Gamma(1+B(\lambda))$ in terms of the Barnes–Vignéras double $\Gamma$–function [V][Sa][UN].

Moreover, $g_k$ is an integer for positive integer $k$, and we establish asymptotic formulas for $g_k$ as $k\to\infty$. We remark that that there are interesting patterns in the prime factorization of $g_k$. The exponents of small primes dividing $g_k$ are quite irregular, and in fact $p\nmid g_{k,U}$ for $k<p<k+\sqrt{p}$. For example,

$$\begin{aligned} g_{100,U} = {} & 2^{95}\cdot 3^{65}\cdot 5^{24}\cdot 7^{33}\cdot 11^{10}\cdot 13^{33}\cdot 17^{36}\cdot 19^{29}\cdot 23^{20}\cdot 29^{16}\cdot 31^{11}\cdot 37^{10}\cdot 41^{12}\\ & \cdot 43^{9}\cdot 47^{4}\cdot 53^{3}\cdot 59^{7}\cdot 61^{9}\cdot 67^{18}\cdot 71^{12}\cdot 73^{10}\cdot 79^{6}\cdot 83^{4}\cdot 89^{2}\cdot 97\cdot 113\\ & \cdot 127^{5}\cdot 131^{7}\cdot 137^{9}\cdot 139^{10}\cdot 149^{16}\cdot 151^{17}\cdot 157^{20}\cdot 163^{24}\cdot 167^{26}\cdot 173^{30}\\ & \cdot 179^{34}\cdot 181^{36}\cdot 191^{43}\cdot 193^{44}\cdot 197^{47}\cdot 199^{47}\cdot 211^{47}\cdot 223^{44}\cdot\ldots\cdot 9973. \end{aligned}$$

Even more interesting are the patterns in the exponent of $p$ in the prime factorization of $g_k$ as $k \to \infty$. We show that there are continuous self–similar functions $c_p(x)$ such that if $k_j = [p^j x]$ then $v_p(g_{k_j,U}) \sim k_j c_p(x)$, and $v_p(g_{k_j,O}) \sim v_p(g_{k_j,Sp}) \sim \frac{1}{2} k_j c_p(x)$, where $v_p(n)$ is the power of $p$ in the factorization of $n$. A graph of $c_3(x)$ is shown here:




Note that these functions satisfy $c_p(x) = c_p(px)$. We will establish other self–similarity properties of $c_p(x)$, and also give an elegant formula for $c_p(x)$.

As a matter of interest, we also report that the Keating–Snaith constant $g_{\frac{1}{2},U}$ can be used to conjecture the mean 1st moment of the $\zeta$–function, and a calculation finds:

$$g_{\frac{1}{2},U} = \frac{\Gamma(\frac{5}{4})\pi^{\frac{1}{4}}}{2^{\frac{1}{6}}} \exp\left(\frac{1}{4}\left(\frac{\zeta'}{\zeta}(2) - \gamma + 1\right)\right) \approx 1.0362329154.... \tag{2.4}$$

It has been shown [CG1][H–B] that $g_{\frac{1}{2},U}$, if it exists, satisfies $1 \le g_{\frac{1}{2},U} \le 16/15$, assuming RH.

The paper is organized as follows. In the next section we discuss the mean square of $L$–functions times a mollifier. The remainder of the paper is devoted to establishing properties of $g_\lambda$ as defined in (2.1)–(2.3). We will write $g_\lambda$ and $B(\lambda)$ to mean $g_{\lambda,X}$ and $B_X(\lambda)$ for $X$ any one of U, O, or Sp. Results which are particular to one of the $g_{\lambda,X}$ will be specifically stated as such. In section 4 we determine analytic properties of $g_\lambda$ as a function of the complex number $\lambda$. This permits us to express $g_\lambda$ in terms of the double $\Gamma$–function. In section 5 we prove integrality properties of $g_k$ for $k$ a positive integer. In section 6 we discuss the self–similar functions $c_p(x)$. In section 7 we give an asymptotic formula for $g_k$ as $k \to \infty$.

## 3. Mollified mean squares

We model our discussion on the mollified mean square required for Levinson's method [Lev][C]. If

$$L(s)^{-1} = \sum_{n=1}^{\infty} \frac{m(n)}{n^s},$$

then our mollifier for $L(s)$ will be

$$M(s,f) = \sum_{n \le y} \frac{m(n)}{n^s} f\left(\frac{\log y/n}{\log y}\right),$$

where $f(x)$ is a real polynomial with $f(0) = 0$, and $y = Y^\theta$ for some $\theta > 0$, where $Y$ will be chosen appropriately for each family. We are interested in an asymptotic formula for the mean square of $L(s)M(s,f)$ for $L$ in the families described in a previous section.

We first consider the Unitary family, for which our model is the Riemann $\zeta$–function. Let

$$\mathcal{M}_U(P,Q,\theta) := \frac{1}{T}\int_1^T \left| Q\left(\frac{-1}{\log T}\frac{d}{da}\right)\zeta(\tfrac{1}{2}+a+it)M(\tfrac{1}{2}+it,P)\right|^2_{a=0} dt.$$

The length of the mollifier is $T^\theta$. The following formula holds [C] for $\theta < \frac{4}{7}$:

$$\mathcal{M}_U(P,Q,\theta) \sim P(1)^2Q(0)^2 + \frac{1}{\theta}\int_0^1\int_0^1 \left(P'(x)Q(y)+\theta P(x)Q'(y)\right)^2 dx\, dy.$$

An interesting simple case is $\mathcal{M}_U(x,1,\theta) \sim 1+\theta^{-1}$.

For the Orthogonal family, the mollifier has length $\sqrt{X}^\theta$. The mollified mean value is:

$$\mathcal{M}_O(P,Q,\theta) = \frac{1}{X^*}\sum_{c(f)\le X}\left(Q\left(\frac{2}{\log X}\frac{d}{da}\right)\xi_f(1/2+a)M(1/2,P)\right)^2\Bigg|_{a=0}.$$

For example applications of related mollified mean values, see [IS][KM][KMV]. If $Q$ is even or odd, then for $\theta < 1$ we have

$$\begin{aligned}\mathcal{M}_O(P,Q,\theta) \sim{} & \left(P(1)Q'(1)+\frac{1}{\theta}P'(1)Q(1)\right)^2 \\ & + \frac{1}{\theta}\int_0^1\int_0^1\left(\frac{1}{\theta}P''(x)Q(y)-\theta P(x)Q''(y)\right)^2 dx\, dy.\end{aligned}$$

In this case we have $\mathcal{M}_O(x,1,\theta) \sim \theta^{-2}$. It seems unexpected that $\mathcal{M}_O(x,1,\theta) \not\to 1$ as $\theta \to 0$. This may be related to the distribution of low–lying zeros of these $L$–functions [ILS].

For the Symplectic family, the mollifier has length $\sqrt{X}^\theta$. The mollified mean value is:

$$\mathcal{M}_{Sp}(P,Q,\theta) := \frac{1}{X^*}\sum_{c(f)\le X}\left(Q\left(\frac{2}{\log X}\frac{d}{da}\right)\xi_f(1/2+a)M(1/2,P)\right)^2\Bigg|_{a=0}.$$

Applications of related mean values can be found in [S]. If $Q$ is odd then $\mathcal{M}_{Sp}(P,Q,\theta) = 0$. If $Q$ is even, then for $\theta < 1$ we have

$$\begin{aligned}\mathcal{M}_{Sp}(P,Q,\theta) \sim{} & \left(P(1)Q(1)+\frac{1}{\theta}P'(1)\hat{Q}(1)\right)^2 \\ & + \frac{1}{\theta}\int_0^1\int_0^1\left(\frac{1}{\theta}P''(x)\hat{Q}(y)-\theta P(x)Q'(y)\right)^2 dx\, dy,\end{aligned}$$

where $\hat{Q}(y) = \int_0^y Q(u)\, du$. We have $\mathcal{M}_{Sp}(x,1,\theta) \sim (1+\theta^{-1})^2$.

We end this discussion by pointing out the beautiful relationship

$$\mathcal{M}_{Sp}(P,Q',\theta) \sim \mathcal{M}_O(P,Q,\theta),$$

which is transparent in the above formulas. It was noted in [S] that this relationship is the source of the amazing "coincidence" of main results in the papers [S] and [KM]. The same "coincidence" appears in the main result of [CGG], and it is plausible that there may be a similar explanation for that connection also.

## 4. Meromorphicity of $g_\lambda$

We prove

**Theorem 4.1.** *The function $g_\lambda/\Gamma(1+B(\lambda))$ is meromorphic of order 2 in the whole complex plane. It never vanishes. Furthermore, for $k = 1, 2, \ldots$,*

*U)* $\dfrac{g_{\lambda,U}}{\Gamma(1+B_U(\lambda))}$ *has a pole of order $2k-1$ at $\lambda = \frac{1}{2} - k$, and no other poles,*

*O)* $\dfrac{g_{\lambda,O}}{\Gamma(1+B_O(\lambda))}$ *has a pole of order $k$ at $\lambda = \frac{1}{2} - k$, and no other poles,*

*Sp)* $\dfrac{g_{\lambda,Sp}}{\Gamma(1+B_{Sp}(\lambda))}$ *has a pole of order $k-1$ at $\lambda = \frac{1}{2} - k$, and no other poles.*

The zero and pole locations of $g_\lambda/\Gamma(1+B(\lambda))$ indicate a connection with the $\Gamma$ and $\Gamma_2$–function. By combining Theorem 4.1 with the asymptotic formulas for $g_k$ given in Theorem 7.1, we obtain the following

**Corollary 4.2.** *We have the following representations of $g_\lambda$ in terms of the $\Gamma$–function and the Barnes–Vignéras double $\Gamma$–function:*

*U)* $$\frac{g_{\lambda,U}}{\Gamma(1+B_U(\lambda))} = 2^{\frac{1}{12}}\, e^{3\zeta'(-1)}\, e^{-2\lambda\zeta'(0)}\, 2^{-2\lambda^2}\, \frac{\Gamma_2(\lambda+\frac{1}{2})^2}{\Gamma(\lambda+\frac{1}{2})},$$

*O)* $$\frac{g_{\lambda,O}}{\Gamma(1+B_O(\lambda))} = 2^{-\frac{17}{24}}\, e^{\frac{3}{2}\zeta'(-1)+\frac{1}{2}\zeta'(0)}\, e^{-\lambda\zeta'(0)}\, 2^{\lambda}\, 2^{-\frac{1}{2}\lambda^2}\, \Gamma_2(\lambda+\frac{1}{2}),$$

*Sp)* $$\frac{g_{\lambda,Sp}}{\Gamma(1+B_{Sp}(\lambda))} = 2^{-\frac{5}{24}}\, e^{\frac{3}{2}\zeta'(-1)-\frac{1}{2}\zeta'(0)}\, e^{-\lambda\zeta'(0)}\, 2^{-\lambda}\, 2^{-\frac{1}{2}\lambda^2}\, \frac{\Gamma_2(\lambda+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})}.$$

*In particular, $g_{\lambda+1,O} = 2^\lambda g_{\lambda,Sp}$, and*

$$\frac{g_{\lambda,O}}{\Gamma(1+B_O(\lambda))}\frac{g_{\lambda,Sp}}{\Gamma(1+B_{Sp}(\lambda))} = 2^{\lambda^2-1}\frac{g_{\lambda,U}}{\Gamma(1+B_U(\lambda))}.$$

See [V][UN] for details about the double $\Gamma$–function.

*Proof of Theorem 4.1.* We illustrate with the case of $g_{\lambda,U}$, the other cases being similar. Choose $J$ such that $1+|2\lambda| < J < |4\lambda|$. Then, for $j \geq J$ the real part of $j+2\lambda$ is positive. We will show that

$$\lim_{N\to\infty}\left(-\lambda^2\log N+\sum_{j=J}^{N}\log\Gamma(j)-2\log\Gamma(j+\lambda)+\log\Gamma(j+2\lambda)\right) \tag{4.1}$$

exists and is bounded by $|\lambda|^{2+\epsilon}$ for any $\epsilon > 0$. This assertion, together with well–known properties of the $\Gamma$–function, imply that $g_{\lambda,U}/\Gamma(1+B_U(\lambda))$ is meromorphic of order at most 2, with zeros and poles as described. It follows that the order of $g_{\lambda,U}/\Gamma(1+B_U(\lambda))$ is exactly 2 because the series $\sum \rho^{-r}$, where $\rho$ runs through the set of poles, with multiplicity, has exponent of convergence $r = 2$.

Now we prove the above assertion. We need to estimate

$$\sum_{j=J}^{N} f_j(0)-2f_j(\lambda)+f_j(2\lambda),$$

where $f_j(x) = \log\Gamma(x+j)$. By a mean value theorem,

$$f_j(0)-2f_j(\lambda)+f_j(2\lambda)=\lambda^2 f_j''(\xi_j)$$

for some $\xi_j$ between 0 and $2\lambda$.

To evaluate $f_j''(\xi_j)$, use the following well–known formula (see [R]), valid for $s \neq 0$ and not on the negative real axis,

$$\log\Gamma(s)=(s-1/2)\log s-s+\frac{1}{2}\log 2\pi+\frac{1}{2}\int_0^\infty \frac{\psi_2(u)+\frac{1}{6}}{(u+s)^2}\,du$$

with

$$\psi_2(u)=\{u\}^2-\{u\}$$

and $\{u\} = u-[u]$. We have

$$\begin{aligned} f_j''(\xi) &= \frac{1}{\xi+j}-\frac{1}{2(\xi+j)^2}-6\int_0^\infty \frac{\psi_2(u)}{(u+j+\xi)^4}\,du \\ &= \frac{1}{j}+O\left(\xi/j^2\right). \end{aligned}$$

It follows that the limit in (4.1) exists and is bounded by $\lambda^2\log J+O(\lambda/J) \ll 1+\lambda^{2+\varepsilon}$. All the assertions are proven.

## 5. Integrality results

In this section $k$ is a positive integer and we will be interested in integrality properties of $g_k$. By virtue of Corollary 4.2, these can be used to deduce rationality results for the double $\Gamma$–function. Denote by $v_p(n)$ the power of $p$ dividing $n$.

**Theorem 5.1.** *For all prime $p$ we have $v_p(g_k) \geq 0$, so $g_k$ is an integer. If $p > B(k)$ then $v_p(g_k) = 0$. If $p < B(k)$ then*

$$v_p(g_k) = 0 \text{ if and only if } p^2 > B(k) \text{ and } \begin{cases} k < p < k + \sqrt{p} & (U) \\ k - \sqrt{k+p} < p < k + \sqrt{k+p} & (O), \end{cases}$$

*with the Sp case following from the O case and the relationship $g_{k+1,O} = 2^k g_{k,Sp}$.*

We have the following useful formulas for $g_k$.

**Lemma 5.2.** *For $k$ a positive integer we have*

$$\begin{aligned} g_{k,U} &= B_U(k)!\, \frac{\left(\prod_{j=1}^{k-1} j!\right)^2}{\prod_{j=1}^{2k-1} j!} \\ &= B_U(k)!\; 2^{k-k^2} \prod_{j=1}^{k-1} \frac{1}{(2j-1)!!\,(2j+1)!!} \\ g_{k,O} &= B_O(k)!\; 2^{B_O(k)+k-1} \prod_{j=1}^{k-1} \frac{j!}{2j!} \\ &= B_O(k)!\; 2^{k-1} \prod_{j=1}^{k-1} \frac{1}{(2j-1)!!} \\ g_{k,Sp} &= B_{Sp}(k)!\; 2^{B_{Sp}(k)} \prod_{j=1}^{k} \frac{j!}{2j!} \\ &= B_{Sp}(k)! \prod_{j=1}^{k} \frac{1}{(2j-1)!!}. \end{aligned}$$

Note that relationships between $g_{k,U}$, $g_{k,O}$, and $g_{k,Sp}$, as given at the end of Corollary 4.2, can also be obtained from the above formulas.

*Proof.* The proofs are similar, so we illustrate with the case of $g_{k,U}$. We begin with

formula (2.1), first using the fact that $k$ is an integer.

$$\begin{aligned}
\frac{g_{k,U}}{B_U(k)!} &= N^{-k^2} \lim_{N\to\infty} \prod_{j=1}^{N}\prod_{m=1}^{k} \frac{j+2k-m}{j+k-m} \\
&= \lim_{N\to\infty} \prod_{m=1}^{k} N^{-k} \frac{\prod_{j=N-k+1}^{N}(j+2k-m)}{\prod_{j=1}^{k}(j+k-m)} \\
&= \lim_{N\to\infty} \prod_{m=1}^{k}\prod_{j=1}^{k} N^{-1} \frac{N-j+2k-m}{j+k-m} \\
&= \prod_{m=1}^{k}\prod_{j=1}^{k} \frac{1}{j+k-m} \\
&= \prod_{m=0}^{k-1}\prod_{j=1}^{k} \frac{1}{j+m} \\
&= \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.
\end{aligned}$$

The last expression leads easily to the Lemma.

From Lemma 5.2 it follows that $p \nmid g_k$ for the cases listed in Theorem 5.1. To prove the remaining statements in Theorem 5.1 we will find a precise expression for $v_p(g_k)$. We only consider the cases of $g_{k,U}$ and $g_{k,O}$, the remaining case following immediately from the relationship $g_{k+1,O} = 2^k g_{k,Sp}$. Also, we only consider primes $p > 2$ because it is easy to show that $g_k$ is divisible by a large power of 2. The main tool we need is the following Lemma.

**Lemma 5.3.** *We have*

$$v_p\left(\prod_{j=1}^{J} j!\right) = \sum_{\ell=1}^{\infty} (J+1)\left[\frac{J}{p^\ell}\right] - \frac{p^\ell}{2}\left(\left[\frac{J}{p^\ell}\right]^2 + \left[\frac{J}{p^\ell}\right]\right)$$

*and*

$$v_p\left(\prod_{j=1}^{J} (2j-1)!!\right) = \sum_{\ell=1}^{\infty} \left(J+\frac{1}{2}\right)\left[\frac{2J-1}{p^\ell}\right]_2 - \frac{p^\ell}{2}\left[\frac{2J-1}{p^\ell}\right]_2^2,$$

*where* $[x]_2 = [\frac{1}{2}([x]+1)]$.

*Proof.* Regrouping the products gives

$$\prod_{j=1}^{J} j! = \prod_{j=1}^{J} j^{J-j+1} \qquad \text{and} \qquad \prod_{j=1}^{J} (2j-1)!! = \prod_{j=1}^{J} (2j-1)^{J-j+1}.$$

Thus,

$$v_p\left(\prod_{j=1}^{J} j!\right) = \sum_{\ell=1}^{\infty} \sum_{n=1}^{\left[\frac{J}{p^\ell}\right]} (J - np^\ell + 1),$$

and

$$v_p\left(\prod_{j=1}^{J} (2j-1)!!\right) = \sum_{\ell=1}^{\infty} \sum_{n=1}^{\left[\frac{2J-1}{p^\ell}\right]_2} J + \frac{p^\ell+1}{2} - np^\ell.$$

Evaluating the inner sums gives the Lemma.

Combining Lemmas 5.2 and 5.3 we obtain the following expressions for $v_p(g_k)$.

**Proposition 5.4.** *We have*

$$v_p(g_{k,U}) = \sum_{\ell=1}^{\infty}\Bigg(\left[\frac{k^2}{p^\ell}\right] + (2k-p^\ell)\left[\frac{k-1}{p^\ell}\right] + \left(\frac{p^\ell}{2} - 2k\right)\left[\frac{2k-1}{p^\ell}\right] - p^\ell\left[\frac{k-1}{p^\ell}\right]^2 + \frac{p^\ell}{2}\left[\frac{2k-1}{p^\ell}\right]^2\Bigg)$$

*and*

$$v_p(g_{k,O}) = \sum_{\ell=1}^{\infty}\left[\frac{\frac{1}{2}k(k-1)}{p^\ell}\right] - \left(k-\frac{1}{2}\right)\left[\frac{2k-3}{p^\ell}\right]_2 + \frac{p^\ell}{2}\left[\frac{2k-3}{p^\ell}\right]_2^2.$$

To complete the proof we must obtain lower bounds for the above expressions. The $g_{k,O}$ case is slightly simpler, so we handle it first.

Denote the summand in the second statement of Proposition 5.4 by $v_{p,\ell}(g_{k,O})$. Use $\theta$ to denote a number $0 \le \theta < 1$. For example, $[x] = x - \theta$. We have:

$$\begin{aligned} v_{p,\ell}(g_{k,O}) &= \left[\frac{\frac{1}{2}k(k-1)}{p^\ell}\right] - \left(k-\frac{1}{2}\right)\left[\frac{2k-3}{p^\ell}\right]_2 + \frac{p^\ell}{2}\left[\frac{2k-3}{p^\ell}\right]_2^2 \\ &= \frac{1}{2p^\ell}\left(k^2-k\right) - \theta - \left(k-\frac{1}{2}\right)\left[\frac{2k-3}{p^\ell}\right]_2 + \frac{p^\ell}{2}\left[\frac{2k-3}{p^\ell}\right]_2^2 \\ &= \frac{1}{2p^\ell}\left(\left(k - p^\ell\left[\frac{2k-3}{p^\ell}\right]_2\right)^2 - \left(k - p^\ell\left[\frac{2k-3}{p^\ell}\right]_2\right)\right) - \theta \qquad (5.1) \\ &> -1, \end{aligned}$$

because $M^2 - M \ge 0$ for integral $M$. Thus, $v_{p,\ell}(g_{k,O}) \ge 0$ for all $p$ and $\ell$, so $g_{k,O}$ is an integer. The remaining statements about $v_p(g_{k,O})$ can be obtained by considering the possibilities for $v_{p,\ell}(g_{k,O})$.

For the case of $g_{k,U}$, denote the summand in the first statement of Proposition 5.4 by $v_{p,\ell}(g_{k,U})$. There are two cases:

*Case 1.* $\left[\frac{2k-1}{p^\ell}\right]=2\left[\frac{k-1}{p^\ell}\right]$. We have

$$\begin{aligned} v_{p,\ell}(g_{k,U}) &= \left[\frac{k^2}{p^\ell}\right]-2k\left[\frac{k-1}{p^\ell}\right]+p^\ell\left[\frac{k-1}{p^\ell}\right]^2 \\ &= \frac{k^2}{p^\ell}-\theta-2k\left[\frac{k-1}{p^\ell}\right]+p^\ell\left[\frac{k-1}{p^\ell}\right]^2 \\ &= p^{-\ell}\left(k-p^\ell\left[\frac{k-1}{p^\ell}\right]\right)^2-\theta. \qquad (5.2) \\ &> -1. \end{aligned}$$

Thus, $v_{p,\ell}(g_{k,U})\ge 0$.

*Case 2.* $\left[\frac{2k-1}{p^\ell}\right]=2\left[\frac{k-1}{p^\ell}\right]+1$. We have

$$\begin{aligned} v_{p,\ell}(g_{k,U}) &= \left[\frac{k^2}{p^\ell}\right]+(2p^\ell-2k)\left[\frac{k-1}{p^\ell}\right]+p^\ell\left[\frac{k-1}{p^\ell}\right]^2+p^\ell-2k \\ &= \frac{k^2}{p^\ell}-\theta+(2p^\ell-2k)\left[\frac{k-1}{p^\ell}\right]+p^\ell\left[\frac{k-1}{p^\ell}\right]^2+p^\ell-2k \\ &= p^{-\ell}\left(k-p^\ell-p^\ell\left[\frac{k-1}{p^\ell}\right]\right)^2-\theta. \qquad (5.3) \\ &> -1. \end{aligned}$$

Again, $v_{p,\ell}(g_{k,U})\ge 0$.

This covers all cases because for integral $m$, $n$, we have

$$\frac{n}{m}-1+\frac{1}{m}\le\left[\frac{n}{m}\right]\le\frac{n}{m}+1-\frac{1}{m}.$$

This proves that $g_{k,U}$ is an integer. The remaining statements can be proven by letting $\ell=1$ and considering the possible cases. This completes the proof of Theorem 5.1.

## 6. The function $c_p$

In this section we study the prime factorization of $g_k$ as $k\to\infty$. We find that $v_p(g_k)$ is described by an interesting self–similar function.

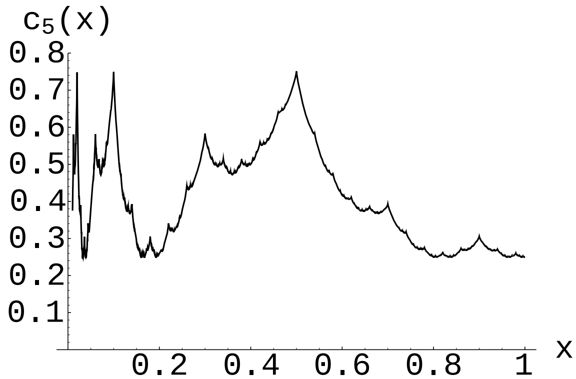
**Theorem 6.1.** *Define*

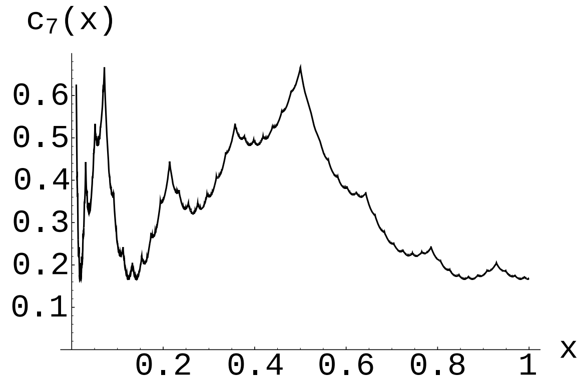
$$c_p(x)=x^{-1}\sum_{\ell=-\infty}^{\infty}p^{-\ell}||p^\ell x||^2,$$

*where* $||x||$ *is the distance from* $x$ *to the nearest integer. Let* $x>0$ *and put* $k_j=[p^jx]$. *Then as* $j\to\infty$,

$$v_p(g_{k_j,U})\sim k_jc_p(x) \qquad \text{and} \qquad v_p(g_{k_j,O})\sim v_p(g_{k_j,Sp})\sim\frac{1}{2}k_jc_p(x).$$

We will leave to the reader the exercise of showing that $c_2(x) = 1$. For the remainder of this section we will assume $p \geq 3$. Here are some graphs:




Note that as $p \to \infty$, $c_p(x)$ approaches $x^{-1}||x||$, uniformly for $\delta \leq x \leq \delta^{-1}$.

From the graphs it appears that each $c_p(x)$ is not differentiable. Much more is actually true. At each rational point $a/b$, the function $c_p(x)$ is either self–similar, or it has a cusp, or it has a vertical tangent. This is made precise in Theorem 6.2.

Let $[[n]]$ denote the absolute least residue of $n \bmod b$. That is, $n \equiv [[n]] \bmod b$ and $b/2 < [[n]] \leq b/2$.

**Theorem 6.2.** *Suppose $p$, $a$, and $b$ are pairwise coprime, and suppose $p^r \equiv 1 \bmod b$, with $r > 0$. Let $f(x) = ||x||^2$.*

*If $\displaystyle\sum_{j=0}^{r-1} [[ap^j]] = 0$, then $c_p(x)$ has the following self–similarity property at $x = a/b$ :*

$$\lim_{\substack{m\to\infty \\ m\equiv m_0 \bmod r}} \frac{c_p\left(\frac{a}{b}+p^{-m}x\right)-c_p\left(\frac{a}{b}\right)}{p^{-m}x} = \frac{x}{p-1} + \sum_{\ell=1-m_0}^{\infty} f'\left(p^{-\ell}\frac{a}{b}\right)$$
$$+ x^{-1}\sum_{\ell=0}^{\infty} p^{-\ell}\left(f\left(p^{\ell+m_0}\frac{a}{b}+p^{\ell}x\right) - f\left(p^{\ell+m_0}\frac{a}{b}\right)\right).$$

*Note that this holds if $b = 1$, or more generally if $b \neq 2$ and $p^n \equiv -1 \bmod b$ for some $n$.*

*Suppose the above condition does not hold. In the case $b = 2$ we have*

$$c_p\left(\frac{a}{2}+p^{-m}x\right) = c_p\left(\frac{a}{2}\right) - mp^{-m}|x| + O(p^{-m}x),$$

*so $c_p$ has a cusp at $a/2$. For all other $b$ there is a nonzero constant $k$ so that*

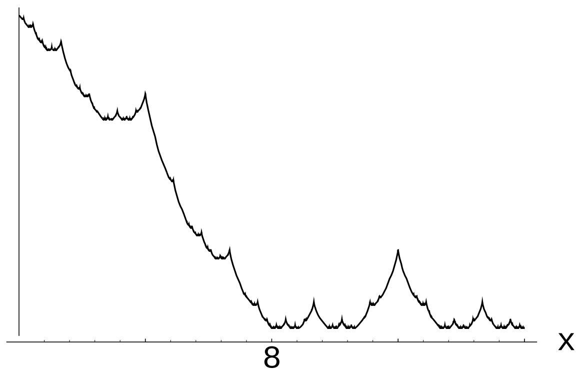
$$c_p\left(\frac{a}{b}+p^{-m}x\right) = c_p\left(\frac{a}{b}\right) + kmp^{-m}x + O(p^{-m}x),$$

*so $c_p$ has a vertical tangent at $a/b$.*

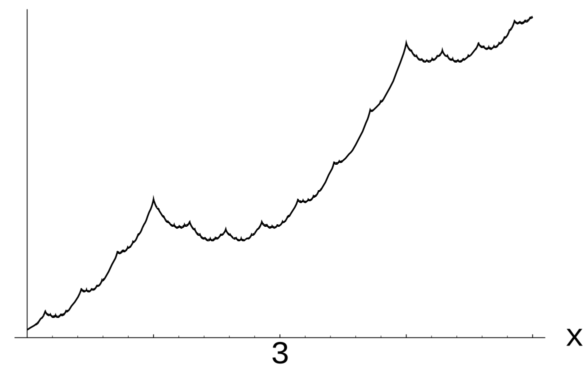
Note: if $(p, b) \neq 1$ then one can use the relation $c_p(x) = c_p(p^j x)$ and then apply the Theorem at $p^j a/b$.

We present some graphs of $c_p(x)$ near rational points $a/b$ where they are self–similar. The simplest case is is when $b = 1$, for then $c_p(x)$ is self–similar under scaling by $1/p$. We give the examples of $c_3(x)$ near $x = 8$ and $c_7(x)$ near $x = 3$. In each graph the $x$–axis is interpreted as extending $1/p^m$ on either side of the central point, for any large $m$, and the vertical scale is also on the order of $1/p^m$.

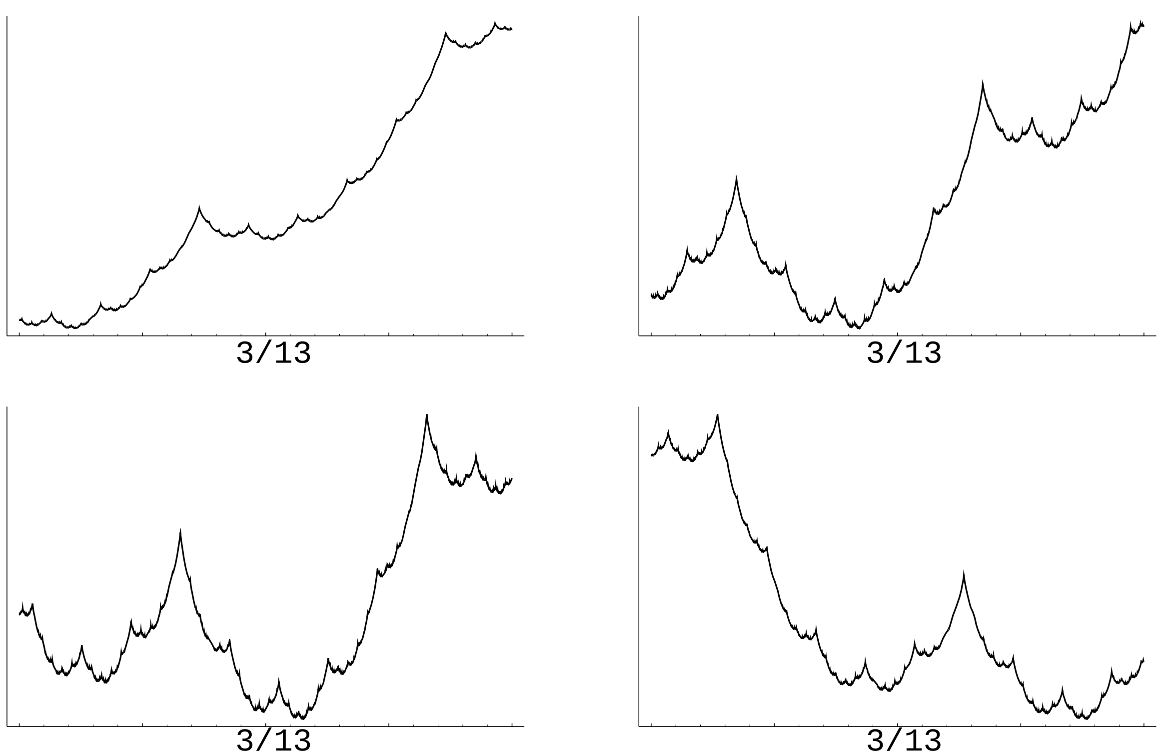
$c_3(x)$ near $x = 8$




$c_7(x)$ near $x = 3$




Our last example is $c_5(x)$ near $x = 3/13$. Since 5 has order $4 \bmod 13$, and $5^2 \equiv -1 \bmod 13$, the function is self–similar on rescaling by $1/p^4$. Thus, we have four possible pictures. In the graphs below, the $x$–axis is interpreted as extending $1/p^m$ on either side of the central point, with $m \equiv i \bmod 4$ for the graph in the $i$th quadrant. The graphs should be read counterclockwise, with each successive graph being the middle 1/5th of the previous graph.



Note that $c_5(3/13) = 23/72$. It is not difficult to evaluate $c_p(a/b)$ for any particular $a/b$, and in addition one sees that $c_p(a/b)$ is rational.

The self–similarity properties of $c_p(x)$ follow from the formula for $c_p(x)$ in Theorem 6.1. We have

**Proposition 6.3.** *Let $f$ be continuous on $\mathbb{R}$, periodic with period $1$, and twice continuously differentiable except at finitely many points $(\bmod\ 1)$ at which it is twice differentiable from both the left and the right. Also suppose $f(x) \ll x^2$ and $f'(x) \ll |x|$ as $x \to 0$. Define*

$$d(x) = \sum_{\ell=-\infty}^{\infty} p^{-\ell} f\left(p^{\ell} x\right).$$

*Then $d(x)$ is continuous on $\mathbb{R}$ and satisfies $d(px) = pd(x)$. Also suppose $p^r \equiv 1 \bmod b$ and $m \equiv m_0 \bmod r$, with $m_0 \geq 0$. If $x$ is sufficiently small then*

$$\begin{aligned} d\left(\frac{a}{b} + p^{-m}x\right) - d\left(\frac{a}{b}\right) = {} & p^{-m}x \sum_{\ell=1-m_0}^{\infty} f'_{\pm}\left(p^{-\ell}\frac{a}{b}\right) \\ & + p^{-m}\sum_{\ell=0}^{\infty} p^{-\ell}\left(f\left(p^{\ell+m_0}\frac{a}{b} + p^{\ell}x\right) - f\left(p^{\ell+m_0}\frac{a}{b}\right)\right) \\ & + (m-m_0)p^{-m}x\frac{1}{r}\sum_{j=0}^{r-1} f'_{\pm}\left(p^{j}\frac{a}{b}\right) \\ & + \frac{1}{2}p^{-2m}x^2\sum_{\ell=0}^{m-1} p^{\ell} f''_{\pm}\left(p^{\ell}\frac{a}{b} + p^{\ell-m}\xi\right) + O(p^{-2m}), \end{aligned}$$

*where $\xi = \xi_\ell$ is between $0$ and $x$. Here the $\pm$ refers to a left– or right–derivative and has the same sign as $x$.*

Note: the condition that $x$ be sufficiently small can be made explicit if one specifies the points at which $f$ is not smooth.

*Proof of Theorem 6.2.* Apply Proposition 6.3 with $f(x) = ||x||^2$. We have

$$f'_{\pm}\left(\frac{a}{b}\right) = \begin{cases} f'(\frac{a}{b}) & \text{if } b \neq 2 \\ \mp 1 & \text{if } b = 2, \end{cases}$$

and $f''_{\pm}(\frac{a}{b}) = 2$ for all $\frac{a}{b}$. This makes the sum over $f''_{\pm}$ explicit. For the second sum over $f'_{\pm}$ note that

$$\sum_{j=0}^{r-1} [[ap^j]] = \frac{b}{2}\sum_{j=0}^{r-1} f'\left(p^j\frac{a}{b}\right).$$

This proves Theorem 6.2.

It remains to prove Theorem 6.1 and Proposition 6.3.

*Proof of Proposition 6.3.* The continuity of $d(x)$ follows from the Weierstrass $M$–test, and the functional equation $d(px) = pd(x)$ follows by changing the summation index.

For the remaining properties, write

$$d\left(\frac{a}{b}\right) - d\left(\frac{a}{b} + p^{-m}x\right) = A + B + C,$$

where we used the definition of $d$ and split the resulting sum into three pieces: $0 \le \ell < m$, $m \le \ell < \infty$, and $\ell < 0$.

We have

$$\begin{aligned} B &= \sum_{\ell=m}^{\infty} p^{-\ell}\left(f\left(p^{\ell}\frac{a}{b} + p^{\ell-m}x\right) - f\left(p^{\ell}\frac{a}{b}\right)\right) \\ &= p^{-m}\sum_{\ell=0}^{\infty} p^{-\ell}\left(f\left(p^{\ell+m}\frac{a}{b} + p^{\ell}x\right) - f\left(p^{\ell+m}\frac{a}{b}\right)\right) \\ &= p^{-m}\sum_{\ell=0}^{\infty} p^{-\ell}\left(f\left(p^{\ell+m_0}\frac{a}{b} + p^{\ell}x\right) - f\left(p^{\ell+m_0}\frac{a}{b}\right)\right). \end{aligned}$$

We did a change of variable, then used the assumption $p^r \equiv 1 \bmod b$, the periodicity of $f$, and the fact that $m_0,\ \ell \ge 0$.

Next,

$$\begin{aligned} C &= \sum_{\ell=1}^{\infty} p^{\ell}\left(f\left(p^{-\ell}\frac{a}{b} + p^{-\ell-m}x\right) - f\left(p^{-\ell}\frac{a}{b}\right)\right) \\ &= \sum_{\ell=1}^{\infty} p^{\ell}\left(f'_{\pm}\left(p^{-\ell}\frac{a}{b}\right)p^{-\ell-m}x + O\left(p^{-2\ell-2m}f''_{\pm}\left(p^{-\ell}\frac{a}{b}\right)\right)\right) \\ &= p^{-m}x\sum_{\ell=1}^{\infty} f'_{\pm}\left(p^{-\ell}\frac{a}{b}\right) + O\left(p^{-2m}\right). \end{aligned}$$

We used Taylor's theorem and the fact that $f''$ is bounded.

Next,

$$\begin{aligned} A &= \sum_{\ell=0}^{m-1} p^{-\ell}\left(f\left(p^{\ell}\frac{a}{b} + p^{\ell-m}x\right) - f\left(p^{\ell}\frac{a}{b}\right)\right) \\ &= \sum_{\ell=0}^{m-1} p^{-\ell}\left(f'_{\pm}\left(p^{\ell}\frac{a}{b}\right)p^{\ell-m}x + \frac{1}{2}f''_{\pm}\left(p^{\ell}\frac{a}{b} + p^{\ell-m}\xi\right)p^{2\ell-2m}x^2\right) \\ &= p^{-m}x\sum_{\ell=0}^{m-1} f'_{\pm}\left(p^{\ell}\frac{a}{b}\right) + \frac{1}{2}x^2p^{-2m}\sum_{\ell=0}^{m-1} p^{\ell}f''_{\pm}\left(p^{\ell}\frac{a}{b} + p^{\ell-m}\xi\right), \end{aligned}$$

where $\xi = \xi_\ell$ is between 0 and $x$. This is valid if $x$ is sufficiently small, in terms of $a/b$, $p$,

and $f$. Now,

$$\begin{aligned}\sum_{\ell=0}^{m-1} f'_{\pm}\left(p^{\ell}\frac{a}{b}\right) &= \sum_{j=0}^{r-1}\sum_{\ell=0}^{\frac{m-m_0}{r}-1} f'_{\pm}\left(p^{r\ell+j}\frac{a}{b}\right) + \sum_{j=0}^{m_0-1} f'_{\pm}\left(p^{m-m_0+j}\frac{a}{b}\right)\\ &= \sum_{j=0}^{r-1}\sum_{\ell=0}^{\frac{m-m_0}{r}-1} f'_{\pm}\left(p^{j}\frac{a}{b}\right) + \sum_{j=0}^{m_0-1} f'_{\pm}\left(p^{j}\frac{a}{b}\right)\\ &= \frac{m-m_0}{r}\sum_{j=0}^{r-1} f'_{\pm}\left(p^{j}\frac{a}{b}\right) + \sum_{j=0}^{m_0-1} f'_{\pm}\left(p^{j}\frac{a}{b}\right).\end{aligned}$$

This completes the proof of Proposition 6.3.

*Proof of Theorem 6.1.* First we consider the Unitary case. We may unify formulas (5.2) and (5.3) to have

$$v_p(g_{k,U}) = \sum_{\ell=1}^{\infty} p^{-\ell}\left(k - p^{\ell}\left(\left[\frac{2k-1}{p^{\ell}}\right] - \left[\frac{k-1}{p^{\ell}}\right]\right)\right)^2 + O\left(\log k\right).$$

We have

$$\begin{aligned}\lim_{j\to\infty}\frac{v_p(g_{[p^j x],U})}{p^j} &= \lim_{j\to\infty}\frac{1}{p^j}\sum_{\ell=1}^{\infty} p^{-\ell}\left([p^j x] - p^{\ell}\left(\left[\frac{2[p^j x]-1}{p^{\ell}}\right] - \left[\frac{[p^j x]-1}{p^{\ell}}\right]\right)\right)^2\\ &= \lim_{j\to\infty}\frac{1}{p^j}\sum_{\ell=1}^{\infty} p^{-\ell}\left(p^j x - \theta_j - p^{\ell}\left(\left[\frac{2p^j x - 2\theta_j - 1}{p^{\ell}}\right] - \left[\frac{p^j x - 1}{p^{\ell}}\right]\right)\right)^2\\ &= \lim_{j\to\infty}\frac{1}{p^j}\sum_{\ell=1}^{\infty} p^{\ell}\Big(p^{j-\ell}x - \left[2p^{j-\ell}x - (2\theta_j+1)p^{-\ell}\right] + \left[p^{j-\ell}x - p^{-\ell}\right]\Big)^2\\ &= \lim_{j\to\infty}\sum_{\ell=-\infty}^{j-1} p^{-\ell}\Big(p^{\ell}x - \left[2p^{\ell}x - (2\theta_j+1)p^{\ell-j}\right] + \left[p^{\ell}x - p^{\ell-j}\right]\Big)^2.\end{aligned}$$

We first used $[p^j x] = p^j x - \theta_j$ with $0 \le \theta_j < 1$, and $[[x]/n] = [x/n]$. We eliminated the first $\theta_j$ by multiplying out and verifying that the other terms make no contribution. Then we changed variables $\ell \mapsto j - \ell$

Let

$$a_{p,j}(x) = \sum_{\ell=-\infty}^{j-1} p^{-\ell}\Big(p^{\ell}x - \left[2p^{\ell}x - (2\theta_j+1)p^{\ell-j}\right] + \left[p^{\ell}x - p^{\ell-j}\right]\Big)^2$$

denote the expression inside the above limit.

Let $\lfloor x \rfloor$ denote the largest integer strictly smaller than $x$. Note that

$$[t-\delta] = \lfloor t \rfloor \qquad \text{if} \qquad 0 < \delta < t - \lfloor t \rfloor.$$

Also, there is always such a $\delta$ because $t \neq \lfloor t \rfloor$ for all $t$. We will use this to simplify $a_{p,j}(x)$.

Let $\varepsilon > 0$ be given and choose $L > 0$ so that $p^{-L} < \varepsilon$. Then choose $\delta > 0$ such that

$$[2p^\ell x - \delta] = \lfloor 2p^\ell x \rfloor \qquad \text{and} \qquad [p^\ell x - \delta] = \lfloor p^\ell x \rfloor$$

for $1 \leq \ell \leq L$. Now choose $J$ so that $3p^{L-J} < \delta$. If $j \geq J$ we have

$$\begin{aligned} a_{p,j}(x) = & \sum_{\ell=-\infty}^{L} p^{-\ell}\Big(p^\ell x - \lfloor 2p^\ell x \rfloor + \lfloor p^\ell x \rfloor\Big)^2 \\ & + \sum_{\ell=L+1}^{j-1} p^{-\ell}\Big(p^\ell x - \big[2p^\ell x - (2\theta_j+1)p^{\ell-j}\big] + \big[p^\ell x - p^{\ell-J}\big]\Big)^2 \\ & = \sum_{\ell=-\infty}^{\infty} p^{-\ell}\Big(p^\ell x - \lfloor 2p^\ell x \rfloor + \lfloor p^\ell x \rfloor\Big)^2 + O\left(\sum_{\ell=L+1}^{\infty} p^{-\ell} + \sum_{\ell=L+1}^{J-1} p^{-\ell}\right) \\ & = \sum_{\ell=-\infty}^{\infty} p^{-\ell}\Big(p^\ell x - \lfloor 2p^\ell x \rfloor + \lfloor p^\ell x \rfloor\Big)^2 + O(p^{-L}), \end{aligned}$$

and we were given that $p^{-L} < \varepsilon$. Thus,

$$\lim_{j\to\infty} a_{p,j}(x) = \sum_{\ell=-\infty}^{\infty} p^{-\ell}\Big(p^\ell x - \lfloor 2p^\ell x \rfloor + \lfloor p^\ell x \rfloor\Big)^2.$$

To finish the proof, note that $(x - \lfloor 2x \rfloor + \lfloor x \rfloor)^2 = (x - [2x] + [x])^2 = ||x||^2$.

For the Orthogonal case, a similar calculation shows

$$\begin{aligned} \lim_{j\to\infty} \frac{v_p(g_{[p^j x],O})}{p^j} &= \frac{1}{2} \sum_{\ell=-\infty}^{\infty} p^{-\ell}\left(p^\ell x - [2p^\ell x]_2\right)^2 \\ &= \frac{1}{2}\, x\, c_p(x), \end{aligned}$$

since $(x - [2x]_2)^2 = ||x||^2$. The same holds in the Symplectic case because $g_{k+1,O} = 2^k g_{k,Sp}$. This completes the proof of Theorem 6.1

## 7. Asymptotics

In this section we determine the asymptotic behavior of $g_k$ for large (integer) $k$. We have

**Theorem 7.1.** *As* $k \to \infty$,

$$\begin{aligned}
\log g_{k,U} &= \log B_U(k)! - k^2 \log k + \left(\frac{3}{2} - 2\log 2\right) k^2 - \frac{1}{12}\log k \\
&\qquad + \frac{1}{12}\log 2 + \zeta'(-1) + O\left(k^{-1}\right) \\
&= k^2 \log k + \left(\frac{1}{2} - 2\log 2\right) k^2 + \frac{11}{12}\log k \\
&\qquad + \frac{1}{12}\log 2 - \zeta'(0) + \zeta'(-1) + O\left(k^{-1}\right), \\
\log g_{k,O} &= \log B_O(k)! - \frac{1}{2}k^2 \log k + \left(\frac{3}{4} - \frac{1}{2}\log 2\right) k^2 + \frac{1}{2}k\log k + \left(\log 2 - \frac{1}{2}\right) k \\
&\qquad - \frac{1}{24}\log k - \frac{17}{24}\log 2 + \frac{1}{2}\zeta'(-1) + O\left(k^{-1}\right) \\
&= \frac{1}{2}k^2 \log k + \left(\frac{1}{4} - \log 2\right) k^2 - \frac{1}{2}k\log k + \left(\frac{3}{2}\log 2 - \frac{1}{2}\right) k + \frac{23}{24}\log k \\
&\qquad - \frac{29}{24}\log 2 + \frac{1}{4} - \zeta'(0) + \frac{1}{2}\zeta'(-1) + O\left(k^{-1}\right), \\
\log g_{k,Sp} &= \log B_{Sp}(k)! - \frac{1}{2}k^2 \log k + \left(\frac{3}{4} - \frac{1}{2}\log 2\right) k^2 - \frac{1}{2}k\log k + \left(\frac{1}{2} - \log 2\right) k \\
&\qquad - \frac{1}{24}\log k - \frac{5}{24}\log 2 + \frac{1}{2}\zeta'(-1) + O\left(k^{-1}\right) \\
&= \frac{1}{2}k^2 \log k + \left(\frac{1}{4} - \log 2\right) k^2 + \frac{1}{2}k\log k + \left(\frac{1}{2} - \frac{3}{2}\log 2\right) k + \frac{23}{24}\log k \\
&\qquad - \frac{17}{24}\log 2 + \frac{1}{4} - \zeta'(0) + \frac{1}{2}\zeta'(-1) + O\left(k^{-1}\right).
\end{aligned}$$

Combining these with the asymptotics of the $\Gamma$ and $\Gamma_2$–function [V][UN] we obtain the expressions for $g_\lambda$ in terms of $\Gamma_2$ given in Corollary 4.2.

To prove Theorem 7.1, combine the expressions for $g_\lambda$ given in Lemma 5.1 with the elementary formulas

$$\sum_{j=1}^{n} \log j! = (n+1)\sum_{j=1}^{n} \log j - \sum_{j=1}^{n} j\log j,$$

and

$$\begin{aligned}
\sum_{j=1}^{n} \log 2j! = \frac{1}{2}n(n+1)\log 2 + (n+1)\sum_{j=1}^{n}\log j + (n+1)\sum_{j=1}^{n}\log(2j-1) \\
- \sum_{j=1}^{n} j\log j - \sum_{j=1}^{n} j\log(2j-1).
\end{aligned}$$

The formulas in the following Lemma are sufficient to complete the calculations.

**Lemma 7.2.** *As $n \to \infty$ we have*

$$\sum_{j=1}^{n} \log j = n \log n - n + \frac{1}{2}\log n - \zeta'(0) + \frac{1}{12n} + O\left(n^{-2}\right)$$

$$\sum_{j=1}^{n} \log(2j-1) = n \log 2n - n + \frac{1}{2}\log 2 - \frac{1}{24n} + O\left(n^{-2}\right)$$

$$\sum_{j=1}^{n} j \log j = \frac{1}{2}n^2 \log n - \frac{1}{4}n^2 + \frac{1}{2}n \log n + \frac{1}{12}\log n + \frac{1}{12} - \zeta'(-1) + O(n^{-1})$$

$$\sum_{j=1}^{n} j \log(2j-1) = \frac{1}{2}n^2 \log 2n - \frac{1}{4}n^2 + \frac{1}{2}n \log 2n - \frac{1}{2}n - \frac{1}{24}\log n + \frac{7}{24}\log 2 - \frac{1}{24} + \frac{1}{2}\zeta'(-1) + O(n^{-1})$$

The first formula in Lemma 7.2 is Stirling's formula, and each expression can be derived from the Euler–Maclaurin summation formula (see Rademacher [R]). This completes the proof of Theorem 7.1.

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